Quantum mechanics II, Solution 2 : Entanglement (Part 1)

TA : Achille Mauri, Behrang Tafreshi, Gabriel Pescia, Manuel Rudolph, Reyhaneh Aghaei Saem, Ricard Puig, Sacha Lerch, Samy Conus, Tyson Jones

Sometimes observation kills.

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Problem 1 : Bell States

Consider two spin-1/2 particles A and B with :

$$|\psi(A)\rangle = c_0^{(A)}|0\rangle_A + c_1^{(A)}|1\rangle_A$$

and

$$|\psi(B)\rangle = c_0^{(B)}|0\rangle_B + c_1^{(B)}|1\rangle_B.$$

The states $|0\rangle$ and $|1\rangle$ are the eigenstates of the $\hat{S}_z = \frac{\hbar}{2}\sigma_z$ operator with eigenvalues $+\hbar/2$ and $-\hbar/2$, respectively.

1. Write down the four possible basis states for the composite system $|\psi(A)\rangle \otimes |\psi(B)\rangle$ in terms of the basis vectors $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$.

We seek the basis states of the composite space $\mathcal{H}_A \otimes \mathcal{H}_B$ where $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert spaces of the A and B labelled systems respectively. There are $\dim \mathcal{H}_A \otimes \mathcal{H}_B = \dim \mathcal{H}_A \times \dim \mathcal{H}_B = 2 \times 2 = 4$ such basis states. We can enumerate them by simply tensoring every combination of basis states from the two systems :

$$\{|i\rangle \otimes |j\rangle : |i\rangle \in \text{basis of } \mathcal{H}_A, j \in \text{basis of } \mathcal{H}_B\} \tag{1}$$

These are :

$$|0\rangle_A \otimes |0\rangle_B \tag{2}$$
$$|1\rangle_A \otimes |0\rangle_B \tag{3}$$
$$|0\rangle_A \otimes |1\rangle_B \tag{4}$$
$$|1\rangle_A \otimes |1\rangle_B \tag{5}$$

It is straightforward to see this is an orthonal basis, i.e. $\langle i_{A\otimes B}|j_{A\otimes B}\rangle = \delta_{i,j}$.

2. The Bell states are two-particle states given by :

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B\right)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B\right)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B \right).$$

Show that these four Bell states form an orthonormal basis of the Hilbert space of the two spins $H = H_A \otimes H_B$.

In order for a set of states $\{|\psi_i\rangle\}$ to form an orthonormal basis, we require that :

1. The norm of all the states are 1, i.e. $\langle\psi_i|\psi_i\rangle = 1$.
2. The states are *orthogonal*, i.e. $\langle\psi_i|\psi_j\rangle_{|i\neq j} = 0$.
3. The states are *complete*, i.e. they *span* the entire space of equal-dimension states, such that any therein can be expressed as a linear combination of basis states : $|\psi\rangle = \sum c_i |\psi_i\rangle$ for $c_i \in \mathbb{C}$.

Let us first show property 1 is satisfied. Consider the norm of $|\Phi_+\rangle$:

$$\langle\Phi^+|\Phi^+\rangle = \frac{1}{2}\left(\langle 00|00\rangle + \langle 11|11\rangle + \langle 00|11\rangle + \langle 11|00\rangle\right) \tag{6}$$

and since $\langle 00|11\rangle = \langle 0|1\rangle \langle 0|1\rangle = 0$, it follows that

$$\langle\Phi^+|\Phi^+\rangle = \frac{1}{2}\left(\langle 00|00\rangle + \langle 11|11\rangle\right) = \frac{1}{2}(1+1) = 1. \tag{7}$$

Similarly, we demonstrate the other states are normalised :

$$\langle\Phi^-|\Phi^-\rangle = \frac{1}{2}(\langle 00|00\rangle + \langle 11|11\rangle - \langle 11|00\rangle - \langle 00|11\rangle) = \frac{1}{2}(\langle 00|00\rangle + \langle 11|11\rangle) = 1 \tag{8}$$

$$\langle\Psi^+|\Psi^+\rangle = \frac{1}{2}(\langle 10|10\rangle + \langle 01|01\rangle + \langle 10|01\rangle - \langle 01|10\rangle) = \frac{1}{2}(\langle 10|10\rangle + \langle 01|01\rangle) = 1 \tag{9}$$

$$\langle\Psi^-|\Psi^-\rangle = \frac{1}{2}(\langle 10|10\rangle + \langle 01|01\rangle - \langle 10|01\rangle - \langle 01|10\rangle) = \frac{1}{2}(\langle 10|10\rangle + \langle 01|01\rangle) = 1 \tag{10}$$

We can now move to property 2. We will start by showing that $|\Phi^+\rangle$ is orthogonal to all the other elements of the basis.

$$\langle\Phi^-|\Phi^+\rangle = \frac{1}{2}(\langle 00|00\rangle - \langle 11|11\rangle + \langle 00|11\rangle - \langle 11|00\rangle) = \frac{1}{2}(\langle 00|00\rangle - \langle 11|11\rangle) = \frac{1}{2}(1-1) = 0 \tag{11}$$

Similarly for the remaining elements

$$\langle\Psi^+|\Phi^+\rangle = \frac{1}{2}(\langle 10|00\rangle + \langle 10|11\rangle + \langle 01|00\rangle + \langle 01|11\rangle) = 0 \tag{12}$$

$$\langle\Psi^-|\Phi^+\rangle = \frac{1}{2}(-\langle 10|00\rangle - \langle 10|11\rangle + \langle 01|00\rangle + \langle 01|11\rangle) = 0 \tag{13}$$

There is no need to explicitly demonstrate the adjoint products like $\langle\Phi^+|\Psi^+\rangle$, because $\langle\Phi^+|\Psi^+\rangle = \langle\Psi^+|\Phi^+\rangle^* = 0^* = 0$. The remaining states are also orthogonal :

$$\langle\Psi^+|\Phi^-\rangle = \frac{1}{2}(\langle 10|00\rangle - \langle 10|11\rangle + \langle 01|00\rangle - \langle 01|11\rangle) = 0 \tag{14}$$

$$\langle\Psi^-|\Phi^-\rangle = \frac{1}{2}(-\langle 10|00\rangle + \langle 10|11\rangle + \langle 01|00\rangle - \langle 01|11\rangle) = 0 \tag{15}$$

$$\langle\Psi^-|\Phi^+\rangle = \frac{1}{2}(-\langle 10|10\rangle - \langle 10|01\rangle + \langle 01|10\rangle + \langle 01|01\rangle) = \frac{1}{2}(\langle 01|01\rangle - \langle 10|10\rangle) = 0 \tag{16}$$

We have covered all products, of which there are $\binom{4}{2} = 6$.

Finally, we must show that the states are complete. For an arbitrary set of vectors, we can do this by demonstrating they are linearly independent. However, because we have so far demonstrated $N = 4$ vectors are orthogonal in an N-dimensional space ($\dim \mathcal{H}_A \otimes \mathcal{H}_B = \dim \mathcal{H}_A \dim \mathcal{H}_B = 4$), it is already guaranteed that the vectors are complete. Hooray !

3. Are the four Bell states separable ?

A state $|\psi\rangle$ is separable if it can be written as a tensor product of states, i.e. $|\psi\rangle = |\phi\rangle \otimes |\varphi\rangle$. For a bell state (e.g. $|\Phi^+\rangle$) to be separable, we must be able to express it as an instance of a general two-qubit separable state $(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$, for $a, b, c, d \in \mathbb{C}$. But attempting to do so :

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \equiv ac|00\rangle + bd|11\rangle + ad|01\rangle + bc|10\rangle \tag{17}$$

$$\implies \begin{cases} ac = \frac{1}{\sqrt{2}} \\ bd = \frac{1}{\sqrt{2}} \\ ad = 0 \\ bc = 0 \end{cases} \tag{18}$$

reveals there is no satisfying solution ; the first two simultaneous equations require that a, c, b, d are non-zero, whereas the final two require some *are* zero. $|\Psi^+\rangle$ is therefore not separable. A similar demonstration can be performed for the other Bell states.

4. We now consider the Bell state $|\Psi^+\rangle$.

(a) What is the probability of measuring $-\hbar/2$ when measuring the spin $\hat{S}_z^{(B)}$ of particle B ?

We know that $-\hbar/2$ is the eigenvalue of $\hat{S}_z$ corresponding to eigenstate $|1\rangle$. We ergo seek the probability of the rightmost ket (that of particle B) being in the $|1\rangle$ state. To compute this, we define a projector upon this substate

$$\hat{\Pi}^{(B)}_{-\hbar/2} = \mathbb{1} \otimes |1\rangle\langle 1|\,. \tag{19}$$

This non-unitary operator sets all amplitudes of the composite system which do *not* correspond to the B particle being in $|1\rangle$, to zero. It imposes no constraint on the state of the A particle, as per the $\mathbb{1}$ on the left. The probability we seek is the absolute-value-squared-sum of the remaining amplitudes, which we can handily compute as

$$P_{\hat{S}_z^{(B)}}(-\hbar/2|\Psi^+) = \left\langle \Psi^+ \middle| \hat{\Pi}^{(B)}_{-\hbar/2} \middle| \Psi^+ \right\rangle \tag{20}$$

$$= \left(\frac{1}{\sqrt{2}}\langle 01| + \frac{1}{\sqrt{2}}\langle 10|\right)(\mathbb{1} \otimes |1\rangle\langle 1|)\left(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle\right). \tag{21}$$

We have adopted the rather intimidating notation $P_{\hat{O}}(\lambda|\psi)$ to represent the probability of a measurement of operator $\hat{O}$ (upon state $|\psi\rangle$) yielding its eigenvalue λ. Evaluating the above expression is trivial ; we must be careful to apply the operators upon the correct substates, then simplify our Dirac notation. We will leverage that

$$\Big(\mathbb{1} \otimes |1\rangle\langle 1|\Big)|a\rangle|b\rangle = \Big(\mathbb{1}|a\rangle\Big) \otimes \Big(|1\rangle\langle 1|\,|b\rangle\Big) \tag{22}$$

$$= |a\rangle \otimes |1\rangle\langle 1|b\rangle \tag{23}$$

$$= \langle 1|b\rangle\Big(|a\rangle \otimes |1\rangle\Big) \tag{24}$$

where $\langle 1|b\rangle \in \mathbb{C}$ is a scalar we commuted to the front of our state. Behold the power of Dirac notation which lets us arbitrarily reinterpret an outer product upon a ket as a ket and a scalar !

The probability we seek is then

$$P_{\hat{S}_z^{(B)}}(-\hbar/2|\Psi^+) = \left(\frac{1}{\sqrt{2}}\langle 0|\langle 1| + \frac{1}{\sqrt{2}}\langle 1|\langle 0|\right)(\mathbb{1} \otimes |1\rangle\langle 1|)\left(\frac{1}{\sqrt{2}}|0\rangle|1\rangle + \frac{1}{\sqrt{2}}|1\rangle|0\rangle\right) \tag{25}$$

$$= \left(\frac{1}{\sqrt{2}}\langle 0|\langle 1| + \frac{1}{\sqrt{2}}\langle 1|\langle 0|\right)\left(\frac{1}{\sqrt{2}}\mathbb{1}|0\rangle|1\rangle\langle 1|1\rangle + \frac{1}{\sqrt{2}}\mathbb{1}|1\rangle|1\rangle\langle 1|0\rangle\right) \tag{26}$$

$$= \left(\frac{1}{\sqrt{2}}\langle 0|\langle 1| + \frac{1}{\sqrt{2}}\langle 1|\langle 0|\right)\left(\frac{1}{\sqrt{2}}|0\rangle|1\rangle\right) \tag{27}$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle 01|01\rangle = \frac{1}{2}. \tag{28}$$

(b) Let's assume now that the measurement of the spin B gives us $-\hbar/2$. What is the probability of subsequently measuring $+\hbar/2$ for the spin A ?

Recall that projector $\hat{\Pi}^{(B)}_{-\hbar/2}$ zeros all amplitudes *not* corresponding to the measurement outcome of $-\hbar/2$. The state $\hat{\Pi}^{(B)}_{-\hbar/2}\,|\Psi^+\rangle$ (which appeared in our algebra above) is ergo unnormalised :

$$\hat{\Pi}^{(B)}_{-\hbar/2}\,\left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}}\,|01\rangle\,. \tag{29}$$

To write the post-measurement state, we must renormalise the result of the projector (multiplying by some constant η) so that all remaining amplitudes (absolute-value-squared) sum to 1. In the above case, where only a single non-zero amplitude remains, this is trivial ;

$$\eta\;\hat{\Pi}^{(B)}_{-\hbar/2}\,\left|\Psi^+\right\rangle = |01\rangle\,, \tag{30}$$

and we do not even need to find $\eta = \sqrt{2}$. But in general, we can appreciate that after the projector, only amplitudes $\{\alpha_i\}$ which contribute toward $P_{\hat{S}_z^{(B)}}$ remain, which we now require to sum (absolute-value-squared) to 1. We need to scale up these remaining amplitudes by η such that that $\sum_i |\eta\alpha_i|^2 = 1$, which implies

$$|\eta| = \frac{1}{\sqrt{\sum_i |\alpha_i|^2}} = \frac{1}{\sqrt{P_{\hat{S}_z^{(B)}}(-\hbar/2|\Psi^+)}}. \tag{31}$$

We can always pick n to be real and positive ; so we divide the state by the square-root of the *probability* of the measurement outcome. Handy !

In any case, given our post-measurement state is a basis state $|01\rangle = |0\rangle_A\,|1\rangle_B$, we can immediately read off that the probability of measuring $+\hbar/2$ (the eigenvalue associated with state $|0\rangle$) of spin A is *one*. That is, our prior measurement on the B system has collapsed the composite system into a state where subsequent $\hat{S}_z$ measurement on the A system deterministically yields eigenvalue $\hbar/2$.

5. We now consider $|\theta\rangle = \frac{1}{2}|\Phi^+\rangle + \frac{\sqrt{3}}{2}|\Psi^-\rangle$. Is this state correctly normalised ? What is the probability of obtaining $\pm\hbar/2$ upon measuring $\hat{S}_z^{(A)}$ on the first spin ?

Let us first check that the state is properly normalised. We have already established that the constituent states are orthogonal $\langle\Phi^+|\Psi^-\rangle = 0$, and are each correctly normalised, $\langle\Phi^+|\Phi^+\rangle = \langle\Psi^-|\Psi^-\rangle = 1$. Thus we can quickly show that

$$\langle\theta|\theta\rangle = \frac{1}{4}\left\langle\Phi^+\middle|\Phi^+\right\rangle + \frac{3}{4}\left\langle\Psi^-\middle|\Psi^-\right\rangle = 1. \tag{32}$$

Outcome $\hbar/2$ is the eigenvalue of $\hat{S}_z$ corresponding to the $|0\rangle$ basis state. Ergo we seek

$$P_{\hat{S}_z^{(A)}}(\hbar/2|\theta) = = \langle\theta|\Big(\,|0\rangle\!\langle 0|\otimes\mathbb{1}\Big)|\theta\rangle\,. \tag{33}$$

We first express $|\theta\rangle$ in the $\hat{Z}$-basis, i.e. in terms of $\{|0\rangle\,,|1\rangle\}^{\otimes}$, then apply the projector to discard irrelevant amplitudes, before taking the sum of absolute-values-squared of the remaining amplitudes (equivalent to left-applying another bra).

$$|\theta\rangle = \frac{1}{2}|\Phi^+\rangle + \frac{\sqrt{3}}{2}|\Psi^-\rangle \tag{34}$$

$$= \frac{1}{2}\left(\frac{|00\rangle}{\sqrt{2}} + \frac{|11\rangle}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2}\left(\frac{|01\rangle}{\sqrt{2}} - \frac{|10\rangle}{\sqrt{2}}\right) \tag{35}$$

$$\therefore \Big(|0\rangle\langle 0| \otimes \mathbb{1} \Big) |\theta\rangle = \frac{1}{2\sqrt{2}}\,|00\rangle + \frac{3}{2\sqrt{2}}\,|01\rangle \tag{36}$$

$$\therefore \langle\theta| \Big(|0\rangle\langle 0| \otimes \mathbb{1} \Big) |\theta\rangle = \frac{1}{2\sqrt{2}}\,\frac{1}{2\sqrt{2}}\,\langle 00|00\rangle + \frac{\sqrt{3}}{2\sqrt{2}}\,\frac{\sqrt{3}}{2\sqrt{2}}\,\langle 01|01\rangle \tag{37}$$

$$= \frac{1}{8} + \frac{3}{8} = \frac{1}{2}. \tag{38}$$

We next seek the probability of the other eigenvalue, $P_{\hat{S}_z^{(A)}}(-\hbar/2|\theta)$. However, because $\pm\hbar/2$ are the only two possible measurement outcomes (because the one-qubit spin operator has a dimension of 2 and ergo has only two eigenvectors), we can write down immediately that :

$$P_{\hat{S}_z^{(A)}}(-\hbar/2|\theta) = 1 - P_{\hat{S}_z^{(A)}}(+\hbar/2|\theta) \tag{39}$$

$$= 1 - \frac{1}{2} \tag{40}$$

$$= \frac{1}{2}. \tag{41}$$

6. What is the probability of obtaining $\pm\hbar/2$ upon measuring $\hat{S}_x^{(A)}$ (the x-axis spin of the first particle) for all of the four Bell states ?

We must be careful not to simply read off the amplitudes of the states, because $|0\rangle$ and $|1\rangle$ are *not* the eigenkets of the $\hat{S}_x$ operator, which are instead :

$$|+\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \quad \text{corresponding to } \lambda = \hbar/2 \tag{42}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle \quad \text{corresponding to } \lambda = -\hbar/2 \tag{43}$$

Fortunately our projector expression will never betray us. The probability of a $\hat{S}_x$ measurement upon particle A of the $|\Phi\rangle$ state producing eigenvalue $\hbar/2$ is given by

$$P_{S_x^{(A)}}(\hbar/2|\Phi^+) = \langle\Phi^+| \Big(|+\rangle\langle +| \otimes \mathbb{1} \Big) |\Phi^+\rangle \tag{44}$$

$$= \left(\frac{\langle 00|}{\sqrt{2}} + \frac{\langle 11|}{\sqrt{2}}\right) \big(|+\rangle\langle +| \otimes \mathbb{1} \big) \left(\frac{|00\rangle}{\sqrt{2}} + \frac{|11\rangle}{\sqrt{2}}\right) \tag{45}$$

$$= \left(\frac{\langle 00|}{\sqrt{2}} + \frac{\langle 11|}{\sqrt{2}}\right) \left(\frac{\langle +|0\rangle\ |+,0\rangle}{\sqrt{2}} + \frac{\langle +|1\rangle\ |+,1\rangle}{\sqrt{2}}\right) \tag{46}$$

$$= \frac{\langle +|0\rangle\ \langle 0|+\rangle}{\sqrt{2}\sqrt{2}}\,\langle 0|0\rangle + \frac{\langle +|1\rangle\ \langle 1|+\rangle}{\sqrt{2}\sqrt{2}}\,\langle 1|1\rangle \tag{47}$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \tag{48}$$

where we were immediately able to write down that $\langle 0|+\rangle = \langle 1|+\rangle = \frac{1}{\sqrt{2}}$, and that $\langle 00|+,1\rangle = \langle 0|+\rangle\,\langle 0|1\rangle = \frac{1}{\sqrt{2}} \times 0 = 0$. The converse outcome is merely

$$P_{S_x^{(A)}}(-\hbar/2|\Phi^+) = 1 - P_{S_x^{(A)}}(\hbar/2|\Phi^+) = \frac{1}{2}. \tag{49}$$

Through similar algebra, we can show the probabilities for all the other Bell states are also $\frac{1}{2}$.

Problem 2 : Composite system of two spin-1/2 particles

Consider the following Hamiltonian operator for two spin-1/2 particles :

$$\hat{H} = \mu_x \hat{S}_x^{(A)} \otimes \hat{S}_x^{(B)} + \mu_y \hat{S}_y^{(A)} \otimes \hat{S}_y^{(B)}$$

where $\hat{S}_x^{(A)}$ and $\hat{S}_y^{(A)}$ are the spin operators for the first spin and $\hat{S}_x^{(B)}$ and $\hat{S}_y^{(B)}$ are the spin operators for the second spin.

1. What are the conditions on the coefficients μ_x and μ_y such that $\hat{H}$ is a valid observable ?

A valid observable is Hermitian, satisfying $\hat{H} = \hat{H}^\dagger$. We ergo require that

$$\begin{aligned}
\mu_x\, \hat{S}_x \otimes \hat{S}_x + \mu_y\, \hat{S}_y \otimes \hat{S}_y &= \left(\mu_x\, \hat{S}_x \otimes \hat{S}_x + \mu_y\, \hat{S}_y \otimes \hat{S}_y\right)^\dagger && (50)\\
&= \mu_x^*\, \hat{S}_x^\dagger \otimes \hat{S}_x^\dagger + \mu_y^*\, \hat{S}_y^\dagger \otimes \hat{S}_y^\dagger && \text{(by linearity)}\\
&= \mu_x^*\, \hat{S}_x \otimes \hat{S}_x + \mu_y^*\, \hat{S}_y \otimes \hat{S}_y && (51)
\end{aligned}$$

because the spin operators are Hermitian, i.e. $\hat{S}_x^\dagger = \hat{S}_x$. We must now solve this equation for $\mu_x, \mu_y \in \mathbb{C}$. It may be tempting to instantiate the spin operators as matrices, e.g. $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and solve the system as a set of simultaneous equations. However, we can instead leverage that the Pauli operators form a basis for all one-qubit operators ; by extension, so too do the spin operators. The operator $\hat{S}_x \otimes \hat{S}_x$ is a tensor of basis states, so is itself a basis state of the composite system. As such, $\hat{S}_x \otimes \hat{S}_x$ and $\hat{S}_y \otimes \hat{S}_y$ are distinct basis states, and are ergo orthogonal. We know from linear algebra that

$$a\,\mathbf{v} + b\,\mathbf{u} = c\,\mathbf{v} + d\,\mathbf{u}, \quad \mathbf{v} \perp \mathbf{u} \quad \implies \quad a = c,\ b = d \tag{52}$$

Our spin-operator coefficients ergo satisfy

$$\begin{cases} \mu_x = \mu_x^*, \\ \mu_y = \mu_y^* \end{cases} \implies \begin{cases} \mu_x \in \mathbb{R}, \\ \mu_y \in \mathbb{R}. \end{cases} \tag{53}$$

2. Write down the matrix elements of $\hat{H}$ in the basis of $\hat{S}_z$ (i.e. $\{|0\rangle, |1\rangle\}^{\otimes 2}$).

We will treat $|0\rangle$ and $|1\rangle$ as the basis of $\hat{S}_z$, so that we can write the spin-1/2 operator matrices as Pauli matrices, neglecting the $\hbar/2$ coefficients. To evaluate $\hat{H}$ as a matrix, we simply substitute the spin operators with their matrix forms, and evaluate the tensor product as the matrix Kronecker product.

$$\begin{aligned}
\hat{H} &= \mu_x\, \hat{S}_x \otimes \hat{S}_x + \mu_y\, \hat{S}_y \otimes \hat{S}_y && (54)\\
&= \mu_x \left(\frac{\hbar}{2}\hat{\sigma}_x\right) \otimes \left(\frac{\hbar}{2}\hat{\sigma}_x\right) + \mu_y \left(\frac{\hbar}{2}\hat{\sigma}_y\right) \otimes \left(\frac{\hbar}{2}\hat{\sigma}_y\right) && (55)\\
&\equiv \mu_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \mu_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} && (56)\\
&= \begin{pmatrix} 0 & 0 & 0 & \mu_x - \mu_y \\ 0 & 0 & \mu_x + \mu_y & 0 \\ 0 & \mu_x + \mu_y & 0 & 0 \\ \mu_x - \mu_y & 0 & 0 & 0 \end{pmatrix}. && (57)
\end{aligned}$$

3. Diagonalize the Hamiltonian in this basis and find its eigenvalues and the corresponding eigenvectors.

There are many techniques to find the eigenvalues and eigenvectors of a generic matrix. Our matrix is infact special; it is *anti-diagonal* which permits us to use the many tricks described here. However, we shall proceed as if $\hat{H}$ was an arbitrary matrix.

Let $a, b, c, d \in \mathbb{C}$. An eigenstate $|\phi\rangle = \begin{pmatrix} a & b & c & d \end{pmatrix}^T$ of the Hermitian matrix $\hat{H}$, with corresponding eigenvalue $\lambda \in \mathbb{R}$, satisfies

$$\hat{H}\,|\phi\rangle = \lambda\,|\phi\rangle \tag{58}$$

$$\therefore \begin{pmatrix} 0 & 0 & 0 & \mu_x - \mu_y \\ 0 & 0 & \mu_x + \mu_y & 0 \\ 0 & \mu_x + \mu_y & 0 & 0 \\ \mu_x - \mu_y & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \tag{59}$$

This is a set of four simultaneous equations which we expect to admit four distinct solutions (the dimension of the composite Hilbert space), which we can find algebraically.

$$\begin{cases} (\mu_x - \mu_y)\,d = \lambda\,a \\ (\mu_x + \mu_y)\,c = \lambda\,b \\ (\mu_x + \mu_y)\,b = \lambda\,c \\ (\mu_x - \mu_y)\,a = \lambda\,d \end{cases}. \tag{60}$$

Let us assume that $|\mu_x| \neq |\mu_y|$ so that we do not have to worry about the null-factor scenario when $\mu_x \pm \mu_y = 0$. We also of course assume $\lambda \neq 0$ to avoid the trivial and irrelevant case of $|\phi\rangle = \mathbf{0}$. These simultaneous equations can be arranged to

$$\begin{cases} a = (\mu_x - \mu_y)\,d/\lambda \\ b = (\mu_x + \mu_y)\,c/\lambda \\ b = \lambda\,c/(\mu_x + \mu_y) \\ a = \lambda\,d/(\mu_x - \mu_y) \end{cases} \tag{61}$$

Note we were careful to only ever divide by λ or $\mu_x \pm \mu_y$, which we know are non-zero. Dividing by a, b, c, d could only lead us to solutions where these variables are assumed non-zero, so we would neglect some valid solutions (and likely fail to find all eigenstates), else we would have committed an invalid division by zero.

The middle two equations together imply

$$(\mu_x + \mu_y)\,c/\lambda = \lambda\,c/(\mu_x + \mu_y) \tag{62}$$

which implies either :

$$c = 0 \implies b = 0 \qquad \text{or} \qquad (\mu_x + \mu_y)^2 = \lambda^2 \implies \lambda = \pm(\mu_x + \mu_y). \tag{63}$$

The outer two equations together imply

$$(\mu_x - \mu_y)\,d/\lambda = \lambda\,d/(\mu_x - \mu_y) \tag{64}$$

which implies either

$$d = 0 \implies a = 0, \qquad \text{or} \qquad (\mu_x - \mu_y)^2 = \lambda^2 \implies \lambda = \pm(\mu_x - \mu_y). \tag{65}$$

These constraints are consistent when we assert $c = b = 0$ and user the latter found eigenvalues, and when we asset $a = d = 0$ and use the former found eigenvalues. We visit these four scenarios in-turn, each of which should allow us to solve for one eigenstate.

— When $c = b = 0$ and $\lambda = \mu_x - \mu_y$, the outer two equations imply

$$(\mu_x - \mu_y)\, d/(\mu_x - \mu_y) = (\mu_x - \mu_y)\, d/(\mu_x - \mu_y) \implies d = d, \tag{66}$$

which tells us d is unconstrained. We can ergo set it to any value we wish, to inform a. In linear algebra, we would sensibly choose $d = 1$ but we have an additional constraint on our eigenstates; that they are normalised (i.e. valid L2 states). As such, we choose $d = \frac{1}{\sqrt{2}} \implies a = \frac{1}{\sqrt{2}}$. Our first eigenstate and corresponding eigenvalue is :

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}^T, \quad \lambda_1 = \mu_x - \mu_y \tag{67}$$

— When $c = b = 0$ and $\lambda = -(\mu_x - \mu_y)$, the outer two equations imply

$$-(\mu_x - \mu_y)\, d/(\mu_x - \mu_y) = -(\mu_x - \mu_y)\, d/(\mu_x - \mu_y) \implies d = d \tag{68}$$

where d is again unconstrained. We arbitarily choose $d = -\frac{1}{\sqrt{2}}$, which yields $a = -d = \frac{1}{\sqrt{2}}$ (making our first coefficient, a, positive). Our second eigenstate and corresponding eigenvalue is :

$$|\phi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix}^T, \quad \lambda_2 = -\mu_x + \mu_y \tag{69}$$

— When $a = d = 0$ and $\lambda = \mu_x + \mu_y$, through identical working as above (now using the middle two equations), we discover

$$|\phi_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}^T, \quad \lambda_3 = \mu_x + \mu_y \tag{70}$$

— When $a = d = 0$ and $\lambda = -\mu_x - \mu_y$, we find

$$|\phi_4\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}^T, \quad \lambda_4 = -\mu_x - \mu_y \tag{71}$$

Our general state $|\phi\rangle = \begin{pmatrix} a & b & c & d \end{pmatrix}^T$ corresponds to basis ordering

$$|\phi\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle\,. \tag{72}$$

Our eigenstates can therefore be expressed as

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle\right) = \left|\Phi^+\right\rangle, \tag{73}$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle - |11\rangle\right) = \left|\Phi^-\right\rangle, \tag{74}$$

$$|\phi_3\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle + |10\rangle\right) = \left|\Psi^+\right\rangle, \tag{75}$$

$$|\phi_4\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle - |10\rangle\right) = \left|\Psi^-\right\rangle. \tag{76}$$

We have found that the eigenstates of $\hat{H} = \mu_x\, \hat{S}_x \otimes \hat{S}_x + \mu_y\, \hat{S}_y \otimes \hat{S}_y$ are the four Bell states!

4. Are the eigenvectors of the Hamiltonian separable between the two particles?

The eigenstates correspond to the Bell states, which we have already established are not separable. Recall that we assumed $|\mu_x| \neq |\mu_y|$. What happens when we relax this assumption?